

Transfer matrix method to study electromagnetic showers

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Abstract :

Transfer matrix method gives information about underlying dynamics of a multifractal. In the present studies, transfer matrix method is applied to multifractal properties of a cherenkov image from which probabilities of electromagnetic components are obtained.

Motivation:

In last decade there have been many studies on the fractal /multifractal nature of extensive air showers (EAS). Most of such studies have been reported for cosmic ray studies [1,2] and to lesser extent [3,4] in γ - ray astronomy. In cosmic ray studies [1,2] multifractal nature of density fluctuations has been experimentally verified and Lipshitz-Holder exponent distribution of EAS has been found to be sensitive parameter to identify the nature of individual EAS. In γ -ray astronomy, it has been shown that cherenkov images can be characterized using multifractal approach. It has been found that cherenkov arrival time [5] is also multifractal in nature.

However, multifractal measures obtained for cherenkov images or cosmic ray densities do not give any information about the underlying dynamics.

Earlier Feigenbaum-Jensen-Procaccia (FJP) recognized this problem in chaos theory. This led them to develop a method [6] which connects multifractal measures with underlying dynamics using thermodynamic approach. In this paper we explore the possibility of using FJP method to study electromag-

netic (EM) showers. For this purpose we use information of D_q versus q curve of a simulated cherenkov image.

For a cascade like EAS, the underlying dynamics means that there are energy splits and probability variations of charged particles and photons. So the idea is, to retrieve some information about energy splits or probabilities or both, from the Cherenkov image using FJP method. Since we are using P-model [7] approach, we will be able to obtain information about probabilities only because in P-model there is assumption of equal split of energies.

FJP Method:

In this section we discuss physical outline of FJP method. The detailed mathematical approach is given in the references [6,7]. The core of FJP method is a transfer matrix. The elements of this transfer matrix are scaling functions of the dynamical process. The scaling functions describe the contraction factors of each interval along each branch. The scaling functions are obtained from the partition function. In general transfer matrix is $\infty \times \infty$ matrix. However, for practical applications mostly 2x2 or sometimes 3x3 matrix is used

For any tree structure, each parent produces number of offsprings. At each level of refinement the number of offsprings are increasing. For any two successive refinement levels, ratio R of the partition functions can be obtained. It has been found that this ratio $R = \lambda(\tau)$ is the leading eigenvalue of the transfer matrix. The characteristic equations of this transfer matrix is given as

$$\lambda^2(\tau) - \lambda(\tau)Tr(T) + DET(T) = 0 \quad (1)$$

where Tr and DET are trace and determinant of a matrix T respectively. By solving equation (1), information about underlying dynamics can be obtained.

Multiplicative processes can be visualized in three ways. In the first case, at each level of refinement there is unequal split but equal probability. Such process is called L-model. In the second case, at each level of refinement there is equal split with unequal probability. This process is known as P-model. In the third case, there is unequal split and unequal probability and process is known as LP-model. Most of the problems have been solved using L or P model. Solutions

of LP-model have been found to be unstable.

Transfer matrix for EM showers:

To study EM showers we consider P-model approach. In P-model rearrangement of probabilities, in the cascade results in a multifractal measure. P-models are preferred over other models when there is no information or data available about the underlying dynamics. The concept of multifractal measures was first conceived in turbulence [8] by using P-model.

For P-model [7] the ratio of partition functions for two successive refinements is

$$\frac{\Gamma^{n+1}(q)}{\Gamma^n(q)} = \frac{\sum_{i=1}^{N_n+1} (P_i^{n+1})^q}{\sum_{i=1}^{N_n} (P_i^n)^q} = R^{-\tau} \quad (2)$$

where P_i^n is the probability in the i th-box for n th level of refinement.

The scaling function σ_p is

$$\sigma_p(\epsilon_{n+1}, \dots, \epsilon_0) = \frac{P(\epsilon_{n+1}, \dots, \epsilon_0)}{P(\epsilon_n, \dots, \epsilon_0)} \delta_{\epsilon_n, \epsilon_n'} \dots \delta_{\epsilon_1, \epsilon_1'} \quad (3)$$

$P(\epsilon_n, \dots, \epsilon_0) = P_i^n$, where ϵ_i gives the location of probability on the path of the tree and δ is Kronecker delta function.

The elements of 2x2 transfer matrix for EM showers are $\sigma_p(00)$, $\sigma_p(01)$, $\sigma_p(10)$ and $\sigma_p(11)$. This is the case of one step memory process. The binary digits 0 and 1 correspond to the left (particle) and right (photon) offspring of the parent. In the next level of refinement there are two digits (00,01,10,11), the first digit denoting the offspring being left or right and the second digit corresponds to parent being left or right.

For a given cherenkov image, $\sigma_p(00)$, $\sigma_p(01)$, $\sigma_p(10)$ and $\sigma_p(11)$ are unknown. Transfer matrix T for a P-model can be written as

$$\begin{pmatrix} \sigma_p(00) & \sigma_p(01) \\ \sigma_p(10) & \sigma_p(11) \end{pmatrix}$$

with the condition

$$\sigma_p(00) + \sigma_p(10) = 1 \quad (4)$$

$$\sigma_p(01) + \sigma_p(11) = 1 \quad (5)$$

σ_p 's correspond to the probability of particles and photons with $\sigma_p(00) \neq \sigma_p(01)$ and $\sigma_p(10) \neq \sigma_p(11)$, meaning unequal probabilities for particles and photons of the same parent. The characteristic equation of the transfer matrix is

$$a^{2q} - [\sigma_p^{-\tau}(00) + \sigma_p^{-\tau}(11)]a^q + [\sigma_p^{-\tau}(00)\sigma_p^{-\tau}(11) - \sigma_p^{-\tau}(01)\sigma_p^{-\tau}(10)] = 0 \quad (6)$$

Results:

For a given cherenkov image whose D_q versus q behaviour is known, $D_{-\infty}$, $D_{+\infty}$ can be calculated. For a P-model

$$D_{\infty} = \frac{[\log(\sigma_p(00))]}{[\log(R^{-1})]} \quad (7)$$

$$D_{-\infty} = \frac{[\log(\sigma_p(11))]}{[\log(R^{-1})]} \quad (8)$$

and equation (6) for $q=0$, can be written as

$$1 - [\sigma_p^{-\tau}(00) + \sigma_p^{-\tau}(11)] + [\sigma_p^{-\tau}(00)\sigma_p^{-\tau}(11) - \sigma_p^{-\tau}(01)\sigma_p^{-\tau}(10)] = 0 \quad (9)$$

For a typical γ -ray initiated simulated cherenkov image corresponding to 50 TeV [3] energy, $D_{-\infty}=1.5$, $D_{\infty}=0.6$ and $D_0=1.0$. Using equations (7) and (8), we obtain the value of $\sigma_p(00)=0.66$ and $\sigma_p(11)=0.34$ and from equation (9), we get the value of $\sigma_p(01)\sigma_p(10)$. Using $\sigma_p(00), \sigma_p(11)$ and $\sigma_p(01)\sigma_p(10)$, equation (6), can be solved for different values of q to obtain $\tau(q)$. The resulting $\tau(q)$ versus q values can be compared with simulated or experimental data. Using equations (4) and (5), we have $\sigma_p(00) = \sigma_p(10) = 0.66$ and $\sigma_p(11) = \sigma_p(01) = 0.34$

Discussion:

Feigenbaum et al [6] called multifractal measures as "static objects". Fractals/ multifractal measures are remnants of a complex underlying dynamics. The connection between the dynamics and the resulting generalized dimensions obtained by Feigenbaum et al was indeed a breakthrough. Chhabra et al [7] investigated FJP method in detail and found that D_q versus q results, obtained using L-model, P-model or LP-model may not always be unique. However, Chhabra et al [7] also concluded that that FJP method will give accurate D_q versus q results if (a) there is proper and independent choice of ratio 'R' (b) there

may be some independent clue for choosing L-model or P-model or LP-model. Thus these two conditions become important when D_q versus q is calculated for comparative studies with experimental or simulated data.

Feigenbaum et al [6] applied FJP method to chaos theory. Chabara et al [7] investigated FJP method and applied it to the study of energy dissipation in turbulence. Batumin and Sergeev [8] applied FJP method to the study of intermittency in hadron collisions.

In EM showers it is well known that tree structure is of binary nature. A γ -ray produces e^\pm pair which initiates particle / photon cascade. So at all levels of refinement there are only two possibilities and ratio $R=2$ will not change. Again in γ -ray initiated showers, it is also well known that there is no loss of energy. For hadron initiated showers we cannot use P-model because at each level of interaction there is energy loss.

The resulting values of two probabilities $P_1=0.66$ and $P_2=0.34$ are obtained from transfer matrix method of cherenkov images. These values are unique because using L-model we get equal probabilities and LP model is inherently unstable. These values of probabilities are very close to the results obtained from Heitler's model. Heitler's model gives a simplified picture of EM showers. However, despite its simplicity it predicts some important features of EM showers which include (a) the proportionality between total number of particles and energy (b) the relationship between shower maxima with energy. Recently Heitler's [10] model has been extended to explain important features of hadron showers.

Conclusions:

In this paper we have obtained probabilities of components of electromagnetic cascade from cherenkov images by using transfer matrix method.

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